Computational Topology and the Unique Games Conjecture

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Dictionary

- **(Unique Games)** The problem $UG(k)$, where $k \in \mathbb{N}$ is a CSP where each domain is $\mathbb{Z}_k$ and each constraint is binary and is a bijection.
- **(Γ-Max-2Lin(A))** Let A be an Abelian group. Γ-Max-2Lin(A) consists of those instances of UG where every variable has A as its domain, and each constraint takes the form $x_i - x_j = c$ for some $c \in A$ (not necessarily the same for all constraints).
- **(Gap Problem)** Given a CSP $\mathcal{P}$, the associated *gap problem* $\mathsf{Gap}\mathcal{P}_{c,s}$ is the promise problem of deciding, given an instance x, whether $OPT(x) \leq s$ or $OPT(x) \geq c$.
- **(Gap preserving reduction)** A polynomial-time *gap-preserving reduction* from $\mathsf{Gap}\mathcal{P}_{c,s}$ to $\mathsf{Gap}\mathcal{Q}_{c',s'}$ (say, both minimization or both maximization problems) is a polynomial-time function f such that $OPT_{\mathcal{P}}(x) \leq s \Rightarrow OPT_{\mathcal{Q}}(f(x)) \leq s'$ and $OPT_{\mathcal{P}}(x) \geq c \Rightarrow OPT_{\mathcal{Q}}(f(x)) \geq c'$. Similarly for $\mathcal{P}$ a maximization and $\mathcal{Q}$ a minimization problems.
- **(UGC-completeness)** A problem $\mathcal{P}$ is *UGC-complete* if there are gap-preserving reductions from $\mathsf{Gap}\mathcal{P}_{\alpha,\beta}$ to $\mathsf{Gap}\mathrm{UG}_{1-\varepsilon,\delta}$ and $\mathsf{Gap}\mathrm{UG}_{1-\varepsilon,\delta}$ to $\mathsf{Gap}\mathcal{P}_{\alpha,\beta}$ such that for any $\alpha < 1$, $\beta > 0$ $\mathsf{Gap}\mathcal{P}_{\alpha,\beta}$ is NP-hard to approximate if and only if UGC holds.
- **(Combinatorial CW-complex X in 2-dim)**(cell-complex for short) 0-dim is just a (finite) set of points, which we denote by $V(X)$. The 1-dim complex corresponds to graphs with multi-edges and loops, we denote the 1-simplices by $E(X)$. The 2-dim complex is a multigraph to which we glue several (full) polygons in such a way that a boundary walk of each of these polygons is glued to a walk in $E(X)$. Edges get mapped to edges (we cannot shrink an edge to a vertex). The set of 2-dim faces is denoted by $F(X)$.
- **(d-chains, $C_d(X, A)$, A Abelian group)** The group of d-chains ($d = 0, 1, 2$) is the group of formal A-linear combinations of d-cells in X. This is isomorphic to the group A^{n_d}, where n_d is the number of d-cells in X ($d = 0$: vertices, $d = 1$: edges, $d = 2$: triangles or 2-cells).
- **(Boundary operator ∂_d)** For a 1-simplex $[i, j]$ the $\partial_1([i, j])$ is a 0-chain $[i] - [j]$, and we extend ∂_1 to a function $C_1(X, A) \to C_0(X, A)$ by A-linearity.
  The operator ∂_2 is a boundary of a 2-cell $[i_1, i_2, \dots, i_\ell]$, which is defined as the 1-cycle $[i_1, i_2] + [i_2, i_3] + [i_3, i_4] + \cdots + [i_{\ell-1}, i_\ell] - [i_1, i_\ell]$, and we extend it to a map $C_2(X, A) \to C_1(X, A)$ by A-linearity.
  The operators $\partial_0 \equiv 0$ and $\partial_3 \equiv 0$ by definition.
- **(d-cycles, $Z_d(X, A)$)** $Z_d(X, A) := \ker \partial_d$, that is, those d-chains that have boundary 0. It is a subgroup of $C_{d-1}(X, A)$.
- **(d-boundaries, $B_d(X, A)$)** The image of the boundary map ∂_{d+1}, that is, those d-chains that are boundaries of some $(d + 1)$-chains. It is a subgroup of $Z_d(X, A)$.

- **(d-th Homology, $H_d(X,A)$)** $H_d(X,A) := Z_d(X,A)/B_d(X,A)$. The quotient identifies those d-cycles that differ only by a d-boundary. $H_0(X,A) \cong A^c$ where c is the number of connected components, and if X is a closed 2-manifold then $H_2(X,A) = A$.
- **(Problem 1-HomLoc)** Given a cell complex X and a 1-cycle $a \in Z_1(X,A)$, determine a 1-cycle a' homologous to a with minimum support among all 1-cycles homologous to a.
- **(d-cochain, $C^d(X,A)$)** A d-cochain on X with coefficients in an Abelian group A is a homomorphism $C_d(X,\mathbb{Z}) \to A$; it is determined by its values on the d-cells of X.
- **(Coboundary operator δ_d)** Given a d-cochain f, its *coboundary* is the function $(\delta_d f) \in C^{d+1}(X,A)$ defined by $(\delta_d f)(\Delta) := f(\partial\Delta)$ for any $(d+1)$-cell Δ, and then extended by linearity (wrt. $\mathbb{Z}$ and A).
- **(d-cocycles, $Z^d(X,A)$)** $Z^d(X,A) := \ker \delta_d$, or equivalently those d-cochains that evaluate to 0 on all d-boundaries. It is a subgroup of $C^d(X,A)$.
- **(d-coboundaries, $B^d(X,A)$)** The image of the d-coboundary map δ_d, that is, those d-cochains that are coboundaries of some $(d-1)$-cochain. It is a subgroup of $Z^d(X,A)$.
- **(d-th Cohomology, $H^d(X,A)$)** $H^d(X,A) := Z^d(X,A)/B^d(X,A)$.
- **(Problem 1-CohoLoc(A))** Given a cell complex X and a 1-cocycle $a \in Z^1(X,A)$, find a cohomologous representative with minimal support.

Conjecture (Khot, Unique Games Conjecture (UGC)). *For all $\varepsilon, \delta > 0$, there exists $k \in \mathbb{N}$ such that $\mathsf{Gap}UG(k)_{1-\varepsilon,\delta}$ is NP-hard.*

Theorems

Lemma (1). *(We can add a linear number of edges/constraints without affecting UGC-hardness)*
For a class $\mathcal{A}$ of graphs, let $UG_{\mathcal{A}}$ denote the Unique Games Problem on graphs from $\mathcal{A}$. Given two classes of graphs $\mathcal{A}, \mathcal{B}$, let $f : \mathcal{A} \to \mathcal{B}$ be a polynomial-time computable function such that for all $G \in \mathcal{A}$, $E(G) \subseteq E(f(G))$ and $|E(f(G)) \setminus E(G)| = O(v)$ where v is the number of vertices in G of degree ≥ 1. If the number of edges added is at most av, then there is a gap-preserving reduction from $UG_{\mathcal{A},1-\varepsilon,\delta}$ to $UG_{\mathcal{B},1-\varepsilon_0,\delta_0}$ where $\varepsilon_0 = \varepsilon + \Delta$ and $\delta_0 = \delta + \Delta$, for any $1 > \Delta > 2\delta a/(1+2\delta a)$ (in particular, with $\Delta \to 0$ as $\delta \to 0$).
In particular, if $UG_{\mathcal{A}}$ is UGC-hard, then so is $UG_{\mathcal{B}}$. The same holds with "UG" everywhere replaced by Max-2Lin or Γ-Max-2Lin.

Observation (10). *(1-CohoLoc is 1-HomLoc on surfaces)*
1-Cohomology Localization on CW complexes that are closed surfaces is equivalent to 1-Homology Localization on CW complexes that are closed surfaces.

Observation (15). *(Γ-Max-2Lin(A) on surfaces is 1-CohoLoc(A))*
Γ-Max-2Lin(A) on a cell decomposition of a surface X is equivalent (under gap-preserving reductions) to 1-CohoLoc(A) on the same cell decomposition of the same surface.

Theorem (16, by Xuong). *A connected graph G has a one-face cellular embedding into a closed orientable surface if and only if there exists a spanning tree T such that every connected component of $G \setminus T$ has an even number of edges.*

Theorem (14). *(1-HomLoc on cell decompositions of closed orientable surfaces is UGC-complete)*
The Unique Games Conjecture holds if and only if for any $\varepsilon, \delta > 0$, there is some $k = k(\varepsilon,\delta)$ such that Gap1-HomLoc$_{1-\varepsilon,\delta}$ on cell decompositions of closed orientable surfaces with coefficients in $\mathbb{Z}_k$ is NP-hard.